Realizable response matrices of multiterminal electrical, acoustic, and elastodynamic networks at a given frequency

Graeme W. Milton
Department of Mathematics, University of Utah, Salt Lake City UT 84112, USA
Pierre Seppecher
Laboratoire d'Analysis Non Linéaire Appliquée et Modélisation,
Université de Toulon et du Var, BP 132-83957 La Garde Cedex, France

Abstract

We give a complete characterization of the possible response matrices at a fixed frequency of n-terminal electrical networks of inductors, capacitors, resistors and grounds, and of n-terminal discrete linear elastodynamic networks of springs and point masses, both in the three-dimensional case and in the two-dimensional case. Specifically we construct networks which realize *any* response matrix which is compatible with the known symmetry properties and thermodynamic constraints of response matrices. Due to a mathematical equivalence we also obtain a characterization of the response matrices of discrete acoustic networks.

Keywords: networks, circuits, multiterminal

1 Introduction

It is well known that composites built from high contrast constituents can have moduli or combinations of moduli which are not usually seen in nature. For example, by combining stiff and compliant phases one can obtain composites with a negative Poisson's ratio, having a high shear modulus but low bulk modulus [1, 2]. More generally one can construct anisotropic composites having any desired positive definite elasticity tensor [3, 4, 5]. Composites have recently been constructed with a negative refractive index, having a negative electrical permittivity and a negative magnetic permeability over some frequency range [6]. They have also been constructed with a negative effective density and with a negative effective stiffness [7, 8] over a range of frequencies. Less well known, though perhaps more interesting, is the fact that the equations describing the

macroscopic behavior of composites built from high contrast constituents can be entirely different from those seen in nature. For example one can obtain materials with macroscopic non-Ohmic, possibly non-local, conducting behavior, even though they conform to Ohm's law at the microscale [9, 10, 11, 12, 13, 14], materials with a macroscopic higher order gradient or non-local elastic response even though they are governed by usual linear elasticity equations at the microscale [15, 16, 5]), materials with non-Maxwellian macroscopic electromagnetic behavior [17], even though they conform to Maxwell's equations at the microscale, and materials with macroscopic behavior outside that of continuum elastodynamics even though they are governed by continuum elastodynamics at the microscale [18]. It is becoming increasingly apparent that the usual continuum equations of physics do not apply to materials with exotic microstructures.

One would really like to be able to characterize the possible macroscopic continuum equations that govern the behavior of materials, including materials with exotic microstructures. A strategy for doing this was developed by Camar-Eddine and Seppecher [13, 5]). Basically, the idea is to first show that one can use a continuum construction to model a discrete network, consisting of nodes (terminals) which are strongly coupled to the continuum matrix and other nodes that are effectively hidden because they occupy vanishingly small volume and are essentially uncoupled with the continuum matrix [alternatively, following the ideas of Milton and Willis [19] these nodes might be in an region of the material that is declared to be hidden, where the behavior of the fields do not influence the chosen macroscopic descriptors]. The next step is to characterize the possible responses of discrete networks, in which one only is interested in the behavior at the terminals. The final step is to characterize the possible continuum limits of these discrete structures. This program was successfully carried out for three-dimensional conductivity [13] and three-dimensional linear elasticity [5], giving a complete characterization of the possible macroscopic equations, under some assumptions such as that the source term does not vary on the microscale, and that the macroscopic descriptor is a single potential (for electrical conductivity) or a single displacement field (for linear elasticity).

Our ultimate goal would be to characterize the possible macroscopic electrodynamic, acoustic, and elastodynamic equations, achievable under the assumption that the microstructure does not vary with time, and also when this assumption is relaxed. A more reachable objective would be to characterize the macroscopic behavior under the assumption that the fields are time harmonic, oscillating at a fixed real frequency ω. This paper is devoted to such a characterization, for discrete dynamical electric networks with grounds, discrete acoustic networks, and discrete elastodynamic networks, anticipating that this will be key to understanding the possible macroscopic limits in continuum systems. Curiously, the characterization of the response tensors in the dynamic case turns out to be easier than in the static case. In the static case, the possible response tensors of n-terminal resistor networks in three-dimensions was essentially characterized by Kirchhoff and is known as the generalized $Y-\Delta$ theorem: any n-terminal network is equivalent to an n-terminal network having no internal nodes and with up to $n(n-1)/2$ resistors connecting the terminal pairs. However, to our knowledge, there is

no such characterization in two-dimensions. One exception is for circular planar resistor networks, where the terminals are at the boundary of a circle, and the network is contained within the circle. For this class of planar network Curtis, Ingerman, and Morrow [20] have completely characterized the possible response matrices. For static n-terminal spring networks Camar-Eddine and Seppecher [5] obtained a complete characterization of the possible response matrices in three-dimensions, but again the two-dimensional case remains an open problem.

2 Electrical Circuits

2.1 The lossless electrical case

To begin with, let us treat the case of an n-terminal network consisting only of capacitors and inductors. An n-terminal network is a set of $n+m$ nodes P_r. Each pair (P_r, P_s) of nodes may be connected by capacitors and/or inductors. The n first nodes, called the terminals of the network, are connected to the exterior. When the terminals P_1, $\ldots$,P_n are respectively submitted to voltages $V_1 e^{-i\omega t}$, $\ldots$,$V_n e^{-i\omega t}$, the complex currents[1] entering the n terminals take the form $iA_1 e^{-i\omega t}/\omega$, $\ldots$, $iA_n e^{-i\omega t}/\omega$. If we denote, in the same way, $iI_{r,s} e^{-i\omega t}/\omega$ the complex current flowing to node r from node s, the linear behavior of the capacitors and inductors connecting the two nodes leads to the relation

$$I_{r,s} = -I_{s,r} = k_{r,s}(V_s - V_r) \tag{2.1}$$

where the coefficient $k_{r,s}$ is $k_{r,s} = 1/L$ for a single inductor while $k_{r,s} = -\omega^2 C$ for a single capacitor, where L is the inductance and C is the capacitance. Of course, $k_{r,s} = 0$ when the two nodes are not connected. So any constant $k_{r,s} \in \mathbb{R}$ is possible. Note that, in this description different nodes simply joined by wires are considered as a single node. We do not allow the terminals to be in this situation (no short circuit).

When an internal node P_r (with $r > n$) is not connected to the ground, Kirchhoff's current law must apply. We have

$$\sum_{s=1}^{n+m} I_{r,s} = 0, \tag{2.2}$$

(in which we set $I_{r,r} = 0$). At each terminal P_r (with $r \leq n$) the same law reads

$$A_r + \sum_{s=1}^{n+m} I_{r,s} = 0. \tag{2.3}$$

But an internal node can be connected to the ground. At such a grounded node P_r, a current can flow toward the ground. Equation (2.2) does not apply anymore and has

[1] We have chosen to keep track of the parameters A_r rather than the currents to unify the mathematics, and make the connections with the discrete elastic models discussed in this paper more transparent.

to be replaced by

$$V_r = 0. \tag{2.4}$$

We only consider circuits for which ω is not a resonance frequency. Then the response $\mathbf{A} = (A_r)_{r=1}^n$ depends in a linear way on the applied voltages $\mathbf{V} = (V_r)_{r=1}^n$: there exists an $n \times n$ matrix $\mathbf{W}$ with real coefficients $W_{r,s}$ such that

$$A_r = \sum_{s=1}^{n} W_{r,s} V_s. \tag{2.5}$$

It is well known that this matrix is symmetric

$$W_{r,s} = W_{s,r}, \tag{2.6}$$

being the Schur complement of the matrix characterizing the response when all nodes are regarded as terminals, which is clearly symmetric.

Our goal is to characterize the set of matrices W which can be obtained as a response matrix of a general (grounded) network but we will also consider two possible restrictions for the networks :

Ungrounded networks : In this case, the network is not connected to the ground and, at each internal node P_r (with $r > n$), equation (2.2) applies. In that case the response to a uniform voltage ($V_1 = V_2 = \ldots = V_n$) is zero and the matrix $\mathbf{W}$ has to satisfy

$$\forall r, \ \sum_{s=1}^{n} W_{r,s} = 0. \tag{2.7}$$

Special grounded networks : For reasons which will become clear in section 2.4, where we treat acoustic networks, we pay particular attention to circuits in which inductors are used only to join ungrounded nodes while capacitors are only used to join an ungrounded node to a grounded one. Owing to (2.4) the grounded nodes are easily eliminated in a first step when computing the response matrix of the circuit and at any node P_r connected to a grounded node with a capacitor with capacitance C_r, equation (2.2) has to be replaced by

$$\sum_{s=1}^{n+m} I_{r,s} = -\omega^2 C_r V_r. \tag{2.8}$$

We say that P_r "has capacitance C_r". Such circuits, we call "special grounded networks", can then be considered as networks of nodes P_r with capacitance C_r only joined by inductors.

The cases of grounded and ungrounded networks are very similar. Indeed, when considering an ungrounded network, we can assume, without loss of generality, that one of

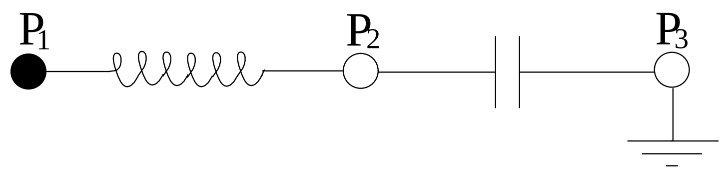

Figure 1: Example 1

the terminals, let say P_n, has voltage 0 and decide to call it the "ground". Due to the constraint (2.7) the response matrix of the network will be determined by the $(n-1)\times(n-1)$ **reduced response matrix** $\widetilde{\mathbf{W}} = (W_{r,s})_{r,s=1}^{n-1}$ where one deletes the n-th row and column from $\mathbf{W}$. Considering P_n as an internal node instead of a terminal, transforms the n-terminal ungrounded network in a $(n-1)$-terminal grounded one. The response matrix of this network coincides with the reduced matrix of the initial network. Reciprocally, when considering a grounded network, it suffices to connect all the grounded nodes together, making so a single node, and to consider this node as a new terminal. We then obtain an ungrounded new network, the reduced matrix of which corresponds to the response matrix of the initial network. The problems of finding all possible response matrices for grounded or ungrounded networks are identical as far as there are no topological or physical restrictions preventing from connecting together all the grounded nodes.

Let us now consider some simple examples of special grounded n-terminal networks. Of course, the same response matrices may be obtained more directly as the response of general grounded networks. We also leave the reader to construct the ungrounded networks with the same reduced response matrix.

Example 1 *Let $k \in \mathbb{R}^*$ (the set of non-zero reals) and consider the simple one-terminal network in which the terminal is connected to an internal node of capacitance $C = \frac{k|k|}{(2k-|k|)\omega^2} > 0$ with an inductor of inductance $L = \frac{2}{|k|}$ (see fig. 1). The response matrix $\mathbf{W}$ is that of the inductor in series with the capacitance:*

$$\mathbf{W} = (\,[L - 1/(\omega^2 C)]^{-1}\,) = (\,k\,)\,. \tag{2.9}$$

Using copies of this circuit with $k > 0$ in a network is equivalent to allowing the use of internal nodes with negative "capacitance" when constructing special grounded networks.

Example 2 *Let $k \in \mathbb{R}^*$ and set $C := \frac{2|k|+k}{\omega^2}$, $L := \frac{1}{2|k|}$, $\widetilde{C} := \frac{4(|k|+k)}{\omega^2}$. We consider the following two-terminal network ($n = 2$): terminals 1 and 2 have capacitance C. They are joined to an internal node P_3 with two inductors of the same inductance L. P_3 has capacitance $\widetilde{C}$ (see fig. 2). We let the reader check that the response matrix of this circuit is the very elementary matrix*

$$\mathbf{W} = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}. \tag{2.10}$$

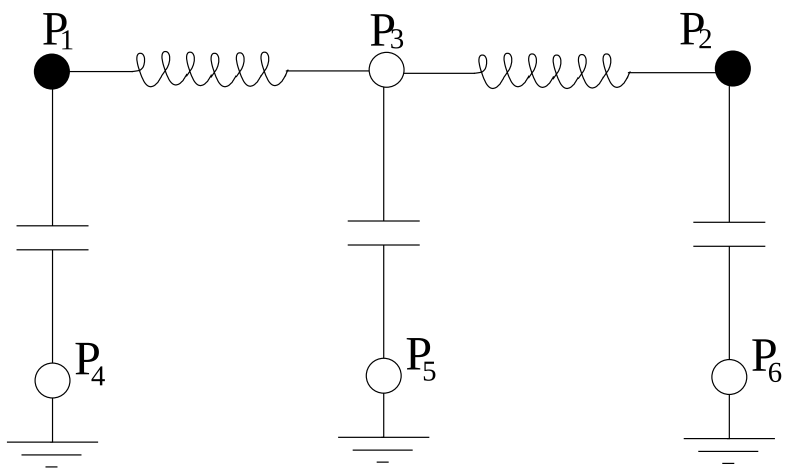

Figure 2: Example 2 and 3

Example 3 *If we modify only the value of C in the previous example by setting $C := \frac{2(|k|+k)}{\omega^2}$, we get the response matrix*

$$\mathbf{W} = \begin{pmatrix} -k & k \\ k & -k \end{pmatrix}. \tag{2.11}$$

When $k < 0$ this response can more directly be obtained by a unique inductor with inductance $-k^{-1}$ joining the two terminals. When $k > 0$ the circuit is equivalent to a unique capacitor with capacitance $k\omega^{-2}$ joining the two terminals.

So, starting with a general grounded network and replacing all capacitors by copies of this circuit leads to a special grounded network with the same response matrix. The restriction to special grounded networks does not reduce the set of possible response matrices.

Superposition principle : We assume that all the components (capacitors, inductors, wires and nodes) the network is made of occupy arbitrarily small volume. Then, in the three-dimensional case, the physical placement of the components has no importance and no crossing problem occurs. When considering two networks sharing the same terminals $(P_1, \ldots, P_n)$, by using (if necessary) a suitable distortion of one of the networks, we can assume that the internal components of the two networks do not intersect. So the response matrix of the network obtained by superposition is simply the sum of the response matrices of the two initial networks (because the two networks share a common set of voltages at the terminals, and the current flowing into each terminal is a sum of the currents flowing into each of the two networks through that terminal). Note that any n-terminal network can be considered as a m-terminal network (with $m > n$) in which $m - n$ terminals are not connected. This superposition property shows also that, in dimension three, the problems of finding all possible response matrices for grounded or ungrounded networks are equivalent.

Then we easily get the following

Theorem 1 *In three-dimensions at a fixed given frequency, any real symmetric matrix* $\mathbf{S}$ *can be realized as the response matrix of a special grounded network. It can also be realized as the reduced response matrix of an ungrounded network.*

Proof: Let n be the dimension of the matrix. Let us construct a special grounded network the response matrix of which is $\mathbf{S}$. We first consider the superposition of n copies of Example 1. The constant k used in the copy attached to terminal r is chosen by setting $k = S_{r,r}$. So the response matrix of the superposition coincides with the diagonal part of $\mathbf{S}$. Then we superimpose on the previous network $\frac{n(n-1)}{2}$ copies of Example 2. The constant k used in the copy attached to the pair of terminals (P_r, P_s) is chosen by setting $k = S_{r,s}$. This fixes the off-diagonal elements of the response matrix.

An ungrounded network, the reduced response matrix of which is $\mathbf{S}$, can be obtained using the correspondence we already described. $\square$

Note that for ungrounded networks a much simpler construction is possible. Given a real $(n+1)$-dimensional symmetric matrix $\mathbf{S}$, whose row sums are zero, we can realize $\mathbf{S}$ as the response matrix of an ungrounded $(n+1)$-terminal network by connecting every pair of terminals (P_r, P_s) with a component with constant $k = -S_{r,s}$: then all the off-diagonal elements take their desired values and the diagonal elements automatically take the correct values by the constraint (2.7).

2.2 The lossy electrical case

Now let us extend our definition of networks by allowing resistors in the connections between nodes. The only change in our analysis is the fact that the constant k in equation (2.1) is no longer real. Indeed for a single resistor with resistance R connecting nodes r and s the relation (2.1) holds with $k = -i\omega/R$. More generally k has a negative imaginary part.

We also slightly extend the definition of special grounded networks by allowing any ungrounded node to be joined to a grounded one by a resistive capacitor. The inductors could also be resistive, but in our constructions we will still require that pairs of ungrounded nodes be connected only by perfect inductors.

The response matrix $\mathbf{W}$ of such circuits is complex, symmetric, with negative semidefinite imaginary part,

$$\mathrm{Im}\mathbf{W} \leq 0. \tag{2.12}$$

This well-known constraint reflects the second law of thermodynamics that the circuit can transform electrical energy into heat but not the reverse. To see this directly it is easy to check that (2.12) is satisfied for a circuit in which all nodes are terminals, and as a result the quantity

$$(\mathrm{Im}\mathbf{V}) \cdot (\mathrm{Re}\mathbf{A}) - (\mathrm{Re}\mathbf{V}) \cdot (\mathrm{Im}\mathbf{A}) = -(\mathrm{Re}\mathbf{V}) \cdot \mathrm{Im}\mathbf{W}(\mathrm{Re}\mathbf{V}) - (\mathrm{Im}\mathbf{V}) \cdot \mathrm{Im}\mathbf{W}(\mathrm{Im}\mathbf{V}) \tag{2.13}$$

(which is proportional to the time averaged power dissipation) is always non-negative, where $\mathbf{V} = (V_1, V_2, \ldots, V_n)$ and $\mathbf{A} = (A_1, A_2, \ldots, A_n) = \mathbf{W}\mathbf{V}$. This remains true if some of the A_r are zero which corresponds to a network with internal nodes. Then the left hand side of (2.13) is just a sum involving the V_r and A_r at the terminals and the algebraic identity implies (2.12) holds for the response matrix $\mathbf{W}$ of a network with internal nodes. It is easy to check that the response matrix has the property (2.12) if and only if the reduced response matrix has a negative semidefinite imaginary part.

We have

Theorem 2 *In dimension three, every symmetric complex matrix* $\mathbf{S}$ *with negative semi-definite imaginary part is realizable as the response matrix of some special grounded network. It is also realizable as the reduced response matrix of an ungrounded network.*

Proof: Let us first reconsider Example 1. Let k be a complex with negative imaginary part and fix $L = \frac{2}{|k|}$ (which is still a positive real and corresponds to a perfect inductor) and $C = \frac{k|k|}{(2k-|k|)\omega^2}$ (which has positive real and imaginary parts and then corresponds to a resistive capacitor). The response matrix is again $\mathbf{W} = (\, k\,)$. Thus we are allowed to use internal nodes with any complex "capacitance" with positive imaginary part when constructing special grounded networks.

Now, let us consider the real and imaginary parts of $\mathbf{S} = \mathbf{S}^{re} + i\mathbf{S}^{im}$. They are $n \times n$ symmetric matrices with real coefficients. Thus we introduce the n eigenvalues $(k^m)_{m=1}^n$ of $\mathbf{S}^{im}$. As $\mathbf{S}^{im}$ is a negative semidefinite symmetric matrix these eigenvalues (k^m) are non-positive reals, and the associated eigenvectors can be chosen to be orthonormal. We denote by $(a_1^m, a_2^m, \ldots a_n^m)$ the n-component eigenvector associated with the eigenvalue k^m.

Owing to Theorem 1 we know that there exists a lossless electrical circuit with $2n$ terminals with the real response matrix $\widetilde{\mathbf{S}}$ with entries $\widetilde{S}_{r,s}$ defined by

$$\widetilde{S}_{r,s} = S^{re}_{r,s}, \text{ if } r \leq n \text{ and } s \leq n, \qquad \widetilde{S}_{r,s} = 0, \text{ if } r > n \text{ and } s > n, \tag{2.14}$$

$$\widetilde{S}_{r,n+m} = \widetilde{S}_{n+m,r} = -k^m\, a_r^m, \text{ if } r \leq n \text{ and } m \in \{1, \ldots n\}. \tag{2.15}$$

Let us endow each terminal P_{n+m} (for $m \in \{1, \ldots n\}$) with the effective capacity $C_{n+m} = -ik^m\omega^{-2}$ and consider all these terminals as internal nodes. At each node P_{n+m} (for $m \in \{1, \ldots n\}$) we have

$$k^m \sum_{s=1}^{n} a_s^m\, V_s = -\omega^2 C_{n+m} V_{n+m} = ik^m\, V_{n+m} \tag{2.16}$$

and, at each terminal P_r, for $r \leq n$:

$$A_r = \sum_{s=1}^{n} S^{re}_{r,s} V_s + \sum_{m=1}^{n} -k^m a_r^m\, V_{n+m}. \tag{2.17}$$

Using the first equation to eliminate the terms involving V_{n+m} in the second equation, we get

$$A_r = \sum_{s=1}^{n} \left(S^{re}_{r,s} + i \sum_{m=1}^{n} k^m a^m_r a^m_s \right) V_s. \tag{2.18}$$

Thus we get the desired response matrix. □

2.3 The construction in two-dimensions

If we think of inductors as coils of wires, then it does not make much physical sense to consider a planar circuit. However metals such as gold or silver or other plasmonic materials can have an electrical permittivity which is close to being real and negative over certain frequency ranges and a rectangular block of such a material can function as an inductor [21, 22].

Now the cases of grounded and ungrounded circuits are quite different. In the first case the ground is freely distributed at any internal node while in the second case only the nodes which can be connected with a fixed terminal may be grounded. In two-dimensions we have the important topological restriction that no two edges are allowed to cross without intersecting at a common node. This would suggest that the superposition principle does not apply. Surprisingly, we still have the following

Theorem 3 *Every symmetric complex matrix* $\mathbf{S}$ *with negative semidefinite imaginary part is realizable as the response matrix of some planar special grounded network. It is also realizable as the reduced matrix of a planar ungrounded network.*

Note that the circuits described in Examples 1, 2 or 3 are planar circuits. Let us add a new example.

Example 4 *We consider a planar four-terminal ungrounded network. The four terminals are numbered clockwise 1, 2, 3 and 4. Let* $k \in \mathbb{R}^*$*. We join the pairs of terminals* (P_1, P_2)*,* (P_2, P_3)*,* (P_3, P_4) *and* (P_4, P_1) *by four identical components with constant* $-k$*. We introduce an internal node* P_5 *and join each terminal to* P_5 *by components with constant* $4k$*. At any terminal* $r \in \{1, 2, 3, 4\}$*, denoting* $\tilde{r}$ *the opposite terminal, equation (2.3) reads*

$$A_r = 4k(V_5 - V_r) - k \sum_{\substack{s=1 \\ s \neq \tilde{r}}}^{4} (V_s - V_r) \tag{2.19}$$

and at node P_5*, equation (2.2) reads*

$$4k \sum_{s=1}^{4} (V_5 - V_s) = 0. \tag{2.20}$$

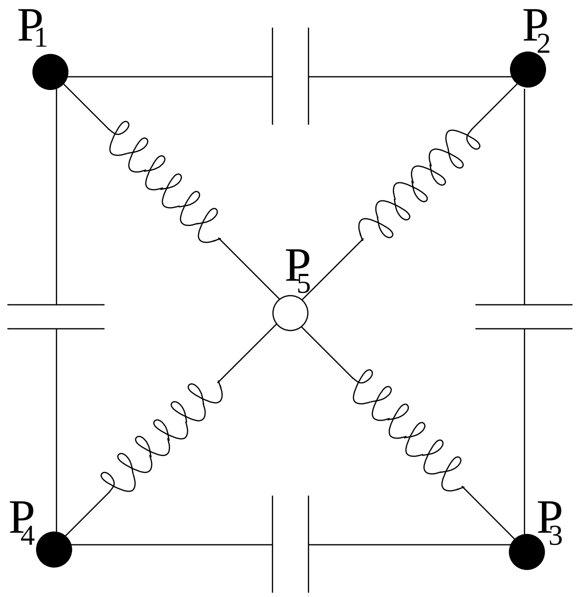

Figure 3: Example 4, the virtual crossing when $k > 0$. When $k < 0$ capacitors and inductors have to be exchanged.

From these two equations we deduce

$$A_r = k \sum_{s=1}^{4} (V_s - V_r) - k \sum_{\substack{s=1 \\ s \neq \tilde{r}}}^{4} (V_s - V_r) = k(V_{\tilde{r}} - V_r) \tag{2.21}$$

The response matrix $\mathbf{W}$ *is*

$$\mathbf{W} = k \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \tag{2.22}$$

Note that, when k is very large this four-terminal circuit is an approximation of a "virtual crossing". The network is equivalent to two connections with constant k joining terminals 1 *and* 3 *and terminals* 2 *and* 4 *without intersecting.*

The same matrix can be obtained as the response matrix of a planar four-terminal special grounded network (under our assumption that the grounds are allowed to be disconnected from each other). Indeed it suffices to replace any capacitor by an ad-hoc copy of Example 3.

Proof of theorem 3. We only consider the case of ungrounded networks. The other case can be treated in a similar way. Theorem 2 provides a three dimensional ungrounded

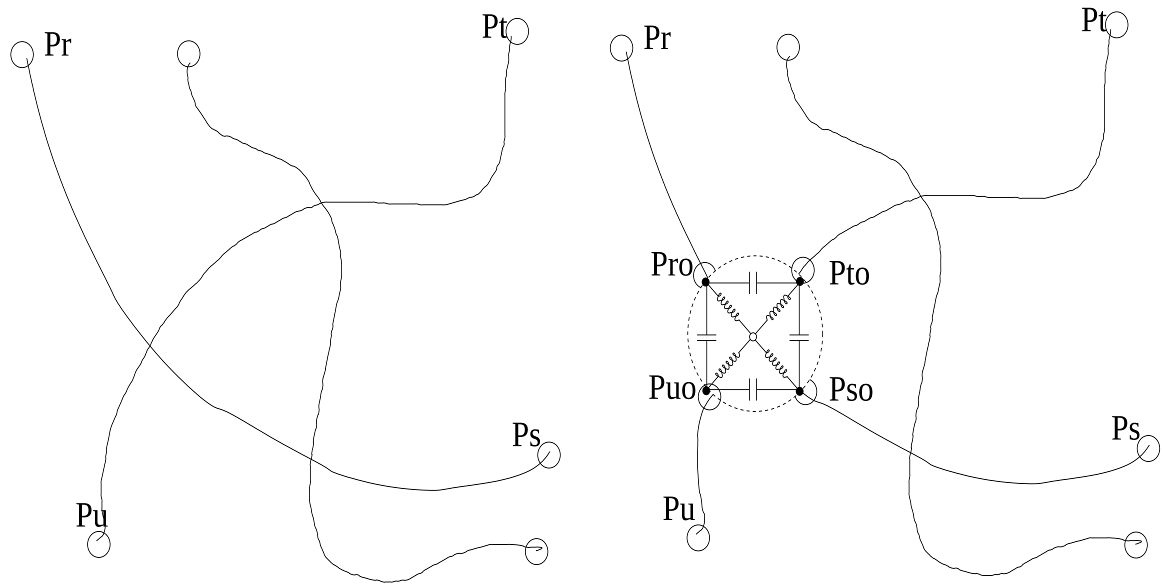

Figure 4: The removal of a crossing.

network which has the desired response matrix. A suitable distorsion transforms this circuit in a planar one. But the resulting network is not a true planar circuit in the sense that the connections between different pairs of nodes (P_r, P_s), (P_t, P_u) cross without any physical interactions. The proof will be completed by proving that any circuit with p crossings is equivalent to another circuit with $p-1$ crossings. So a simple induction argument gives us a planar network without any crossing, that is a true planar circuit.

To remove a crossing point, we use a copy of the network described in Example 4. Let us isolate a particular crossing of two connections (P_r, P_s), (P_t, P_u), the constants of which are denoted respectively $k_{r,s}$ and $k_{t,u}$. We add four internal nodes $P_{r_0}, P_{s_0}, P_{t_0}, P_{u_0}$ and replace the two connections by four connections (P_r, P_{r_0}), (P_s, P_{s_0}), (P_t, P_{t_0}), (P_u, P_{u_0}) and a copy of Example 4 (with an arbitrary constant $k \in \mathbb{R}^*$ satisfying $k \neq k_{r,s}$ and $k \neq k_{t,u}$) as shown in figure 4.

We choose the constants of the connections (P_r, P_{r_0}), (P_s, P_{s_0}), (P_t, P_{t_0}), (P_u, P_{u_0}) respectively equal to

$$k_{r,r_0} = 2\left(\frac{1}{k_{r,s}} - \frac{1}{k}\right)^{-1}, \quad k_{s,s_0} = k_{r,r_0} \tag{2.23}$$

$$k_{t,t_0} = 2\left(\frac{1}{k_{t,u}} - \frac{1}{k}\right)^{-1}, \quad k_{u,u_0} = k_{t,t_0} \tag{2.24}$$

Note that, as $k \in \mathbb{R}^*$, the imaginary parts of these constants are, like the imaginary parts of $k_{r,s}$ and $k_{t,u}$, non-positive. Checking that the response matrix is unchanged is straightforward. $\square$

2.4 Application to the discretized acoustic equation

The preceeding analysis also applies directly to the discretized acoustic equation.

A domestic water supply network is made of tubes containing a (almost) incompressible fluid and some hydraulic capacitors. Such a situation can be found also in natural conditions: for instance in an unsaturated porous medium. Consider a two-dimensional or three-dimensional network of tubes with cavities at the junctions. Each tube contains a segment of incompressible, non-viscous, fluid with some density, possibly varying from tube to tube, moving in a time harmonic oscillatory manner in response to time harmonic pressures at the junctions. (There could an additional time independent constant pressure everywhere, but this does not affect the equations). We define the entire cavity associated with a junction to be the cavity at the junction, plus the remaining region in the tubes not occupied by the incompressible fluid. Each entire cavity contains a compressible, non-viscous, massless fluid with compressibility possibly varying from junction to junction. The surfaces between the compressible and incompressible fluids have some surface energy so that the interfaces remain flat.

When the terminals $P_1, \dots, P_n$ are respectively submitted to pressures $p_1 e^{-i\omega t}, \dots, p_n e^{-i\omega t}$, the complex fluid currents entering the n terminals take the form $iA_1 e^{-i\omega t}/\omega$, $\dots$, $iA_n e^{-i\omega t}/\omega$. We denote, in the same way, $iI_{r,s} e^{-i\omega t}/\omega$ the complex current flowing to node r from node s. Let $a_{r,s}$ be the cross-sectional area of the tube joining nodes P_r and P_s and let $m_{r,s}$ be the mass of the fluid contained in this tube. Since the complex force on the fluid segment is $a_{r,s}(p_r - p_s)$ and its complex acceleration is $I_{r,s} e^{-i\omega t}/a_{r,s}$, Newton's law of motion implies

$$I_{r,s} = -I_{s,r} = k_{r,s}(p_s - p_r) \tag{2.25}$$

where $k_{r,s} = a_{r,s}^2/m_{r,s}$.

Now consider an internal node P_r. Due to the motions of the incompressible fluid segments in the tubes the volume of the associated entire cavity changes with time and the complex pressure p_r in the cavity adjusts itself according to Hooke's law,

$$\sum_{s=1}^{n+m} I_{r,s} = -C_r \omega^2 p_r, \tag{2.26}$$

where $C_r = V_r/\kappa$ in which V_r is the volume of the entire cavity when the fluids are at rest, and κ is the bulk modulus of the fluid in this entire cavity. When the junction is a terminal the sum must take into account the current entering the terminal. If we assume, without loss of generality, that the capacity of each terminal vanishes, we have

$$A_r + \sum_{s=1}^{n+m} I_{r,s} = 0. \tag{2.27}$$

Since (2.25), (2.26) and (2.27) are the direct analogs of equations (2.1) (2.8) and (2.3) which describe special grounded networks, all the previous analysis and associated

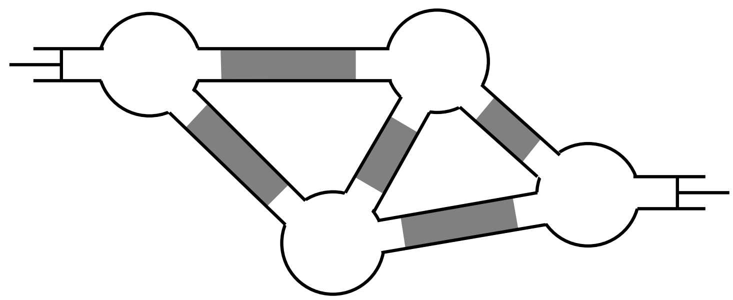

Figure 5: A two-terminal discrete acoustic network. In the idealized model the four cavities contain compressible massless fluid, while the grey shaded fluid plugs in the five tubes contain incompessible fluid with some mass. The response of the network is measured by the movement of the two frictionless pistons, in response to the time harmonic forces acting on them which control the pressures in the terminal cavities.

theorems apply. In particular, to get a negative effective "bulk modulus" in a cavity we just follow the approach of Fang et.al.[8] and connect that cavity to a Helmholtz resonator, i.e. connect it to another cavity containing compressible fluid with a tube containing a plug of incompressible fluid with mass chosen so that the system is above resonance.

The lossy case arises when the fluid in some, or all, of the junctions has some bulk viscosity, so that κ has some negative imaginary part, and consequently C_r in equation (2.26) has a positive imaginary part. The lossy case also arises if the incompressible fluid segments have some shear viscosity so that Darcy's law implies $k_{r,s}$ has a complex part due to the fluid permeability of the tube.

We can conclude that any response is possible for an acoustic discrete system provided that the total dissipation is non-negative.

3 Discrete Elastodynamics

We now turn our attention to networks of springs with point masses at the nodes. We emphasize that our analysis is for idealized linear networks and that we do not consider questions of stability, and in particular stability against buckling.

Let us denote $\mathbf{u}_r e^{-i\omega t}$ the displacement of node P_r ($\mathbf{u}$ is a three dimensional complex

vector) and $\mathbf{F}_{r,s}e^{-i\omega t}$ the force exerted on node P_r by the spring (if any) joining P_s and P_r. In the same way, for any terminal P_r ($r \leq n$), let us denote $\mathbf{A}_r e^{-i\omega t}$ the additional external force applied on the system at that terminal. At each interior node P_r, Newton's law applies and we have

$$\sum_{s=1}^{n+m} \mathbf{F}_{r,s} = -m_r \omega^2 \mathbf{u}_r, \tag{3.1}$$

(which is the analog of (2.8)) where m_r denotes the mass of node P_r. At a terminal P_r ($r \leq n$), we have also to take into account the external force:

$$\mathbf{A}_r + \sum_{s=1}^{n+m} \mathbf{F}_{r,s} = -m_r \omega^2 \mathbf{u}_r. \tag{3.2}$$

3.1 The purely elastic case

To begin with, let us treat the case of purely elastic n-terminal networks where there is no damping in the springs. Between each pair of nodes P_r and P_s, located at positions $\mathbf{x}_r$ and $\mathbf{x}_s$ that are linked by a spring, Hooke's law applies, and we have

$$\mathbf{F}_{r,s} = -\mathbf{F}_{s,r} = k_{r,s}\mathbf{n}_{r,s} \otimes \mathbf{n}_{r,s} \cdot (\mathbf{u}_s - \mathbf{u}_r), \quad \text{where } \mathbf{n}_{r,s} = \frac{\mathbf{x}_s - \mathbf{x}_r}{\|\mathbf{x}_s - \mathbf{x}_r\|}, \tag{3.3}$$

(which is the analog of (2.1)) where $k_{r,s} = k_{s,r}$ is the (positive real) spring constant.

The elastodynamic response of the network is governed by a matrix $\mathcal{W}$ which has second order tensors $\mathbf{W}_{r,s}$ as its entries, and links the set of forces $\mathcal{A} = (\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n)$ with the set of displacements $\mathcal{U} = (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n)$, through the relation

$$\mathbf{A}_r = \sum_{s=1}^{n} \mathbf{W}_{rs}\mathbf{u}_s. \tag{3.4}$$

This matrix $\mathcal{W}$ is real and has the symmetry property that, for any r, s in $\{1, \dots, n\}$,

$$\mathbf{W}_{r,s} = (\mathbf{W}_{s,r})^T. \tag{3.5}$$

and the proof of this property is similar to the proof of (2.6) in the electrical case.

Example 5 *The simplest non-trivial two terminal network just consists of terminals P_1 and P_2 joined by a spring with constant k and direction $\mathbf{n} = \mathbf{n}_{1,2}$. According to (3.3) the matrix $\mathcal{W}$ is*

$$\mathcal{W} = \begin{pmatrix} k\,\mathbf{n}\otimes\mathbf{n} & -k\,\mathbf{n}\otimes\mathbf{n} \\ -k\,\mathbf{n}\otimes\mathbf{n} & k\,\mathbf{n}\otimes\mathbf{n} \end{pmatrix}. \tag{3.6}$$

Example 6 *Let us consider the two terminal network in which the terminals P_1 and P_2 are joined to three internal nodes P_3, P_4, P_5 making two non-degenerate simplexes*

(P_1, P_3, P_4, P_5) and (P_2, P_3, P_4, P_5). To each edge of these simplexes corresponds a spring. Nodes have no mass. In structural mechanics such a structure is called a simple truss. Its response vanishes when the applied displacements $\mathbf{u}_1$, $\mathbf{u}_2$ correspond to a rigid motion. So the response matrix corresponds to a non-negative quadratic form depending only on $(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{n}_{1,2}$. It takes the form (3.6). The constant k can be tuned by multiplying all the constants of the truss by a common positive factor.

Note that the choice of the position of the internal nodes is quite free. A given finite set of points can easily be avoided. Moreover, in dimension three, the internal nodes can be chosen in such a way that the five segments (P_1, P_3), (P_1, P_4) (P_2, P_3), (P_2, P_4) (P_3, P_4) do not intersect a given finite set of straight lines (but maybe at terminals P_1, P_2).

Remark 1 *Replacing a single spring in a network by a structure described in Example 6 will not change the response matrix of the network. In dimension three, making all the needed replacements we can restrict our attention to networks in which any different springs do not intersect and have different directions.*

Example 7 *Let μ be a non vanishing real and $\mathbf{n}$ a unit vector. Consider the very simple one-terminal spring network, where there is only one spring with constant $k = k_{1,2} = \frac{\mu|\mu|}{2\mu-|\mu|}\omega^2$ linking terminal P_1 with a single interior node P_2 chosen in such a way that $\mathbf{n}_{1,2} = \mathbf{n}$. Terminal P_1 has no mass while the mass of node P_2 is $m = \frac{|\mu|}{2}$. We have the equations*

$$\mathbf{F}_{2,1} = k\,\mathbf{n} \otimes \mathbf{n} \cdot (\mathbf{u}_1 - \mathbf{u}_2) = -m\omega^2\,\mathbf{u}_2 = \mathbf{A}_1. \tag{3.7}$$

The elimination of $\mathbf{u}_2$ leads to $\mathbf{A}_1 = -\frac{km\omega^2}{k-m\omega^2}\,\mathbf{n} \otimes \mathbf{n} \cdot \mathbf{u}_1 = -\mu\omega^2\mathbf{n} \otimes \mathbf{n} \cdot \mathbf{u}_1$. This system endows P_1 with the tensor valued "effective mass" : $\mathbf{M} = \mu\mathbf{n} \otimes \mathbf{n}$ which can either be positive or negative semidefinite depending on the sign of μ. The physical reason that one can obtain negative values of μ is that these are achieved when the spring-mass system is above resonance, i.e. when $k - m\omega^2 < 0$, and as a result the mass oscillates $180°$ out of phase with the motion of the terminal.

Note that the choice of the position of the internal node is free on the straight line $(\mathbf{x}_1, \mathbf{n})$. Thus a given finite set of points can easily be avoided. The spring can also be replaced using remark 1 in order to avoid intersections with a given finite set of straight lines.

Now let $\mathbf{M}$ be any real symmetric tensor. Superimposing up to three copies of the previous structure choosing for μ the eigenvalues of $\mathbf{M}$ and for $\mathbf{n}$ the corresponding eigenvectors of $\mathbf{M}$ we get

Remark 2 *Any node can be endowed by any real symmetric tensor effective mass.*

Example 8 *Let K be a real, $\mathbf{n}_1$, $\mathbf{n}_2$ be two unit vectors and $\mathbf{x}_1$, $\mathbf{x}_2$ be two distinct points such that at least one of the two vectors $\mathbf{n}_1$, $\mathbf{n}_2$ is not in the direction $\mathbf{x}_2 - \mathbf{x}_1$. We*

consider the two terminal network consisting of terminals P_1, P_2 at points $\mathbf{x}_1$, $\mathbf{x}_2$ and two internal nodes P_3, P_4 placed in such a way that $\mathbf{n}_{1,3} = \mathbf{n}_1$, $\mathbf{n}_{2,4} = \mathbf{n}_2$ and $\mathbf{v} := \mathbf{n}_{3,4}$ is neither colinear with $\mathbf{n}_1$ nor $\mathbf{n}_2$. We introduce two new unit vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ which complete respectively the basis $(\mathbf{n}_1, \mathbf{v})$ and $(\mathbf{n}_2, \mathbf{v})$.

Springs with constant $k = |K|$ join pairs (P_1, P_3) and (P_2, P_4). A spring with constant $k' = 2|K| - K$ joins (P_3, P_4). Nodes P_3 and P_4 are endowed respectively with the effective masses $k'\omega^{-2}(\mathbf{v} \otimes \mathbf{n}_1 + \mathbf{n}_1 \otimes \mathbf{v} + \mathbf{w}_1 \otimes \mathbf{w}_1)$ and $k'\omega^{-2}(\mathbf{v} \otimes \mathbf{n}_2 + \mathbf{n}_2 \otimes \mathbf{v} + \mathbf{w}_2 \otimes \mathbf{w}_2)$.

As noticed in Example 7 such effective masses need the introduction of extra springs and internal nodes. Again we have a large freedom in the choice of the position of the nodes P_3, P_4 on the lines $(\mathbf{x}_1, \mathbf{n}_1)$ and $(\mathbf{x}_2, \mathbf{n}_2)$ and we can avoid any given finite set of points. Owing to Remark 1 we can also construct this structure avoiding any intersection with a given finite set of straight lines.

Let us introduce the dual basis $(\mathbf{n}_1^*, \mathbf{v}^*, \mathbf{w}_1^*)$ of $(\mathbf{n}_1, \mathbf{v}, \mathbf{w}_1)$ (i.e. satisfying $\mathbf{n}_1^* \cdot \mathbf{n}_1 = 1$, $\mathbf{n}_1^* \cdot \mathbf{v} = 0$, $\mathbf{n}_1^* \cdot \mathbf{w}_1 = 0$, $\mathbf{v}^* \cdot \mathbf{n}_1 = 0$, $\mathbf{v}^* \cdot \mathbf{v} = 1$, $\mathbf{v}^* \cdot \mathbf{w}_1 = 0$, $\mathbf{w}_1^* \cdot \mathbf{n}_1 = 0$, $\mathbf{w}_1^* \cdot \mathbf{v} = 0$ and $\mathbf{w}_1^* \cdot \mathbf{w}_1 = 1$) and in the same way the dual basis $(\mathbf{n}_2^\circ, \mathbf{v}^\circ, \mathbf{w}_2^\circ)$ of $(\mathbf{n}_2, \mathbf{v}, \mathbf{w}_2)$. The displacement of nodes P_3, P_4 are respectively written in the form

$$\mathbf{u}_3 = a\mathbf{n}_1^* + b\mathbf{v}^* + c\mathbf{w}_1^*, \qquad \mathbf{u}_4 = d\mathbf{n}_2^\circ + e\mathbf{v}^\circ + f\mathbf{w}_2^\circ. \tag{3.8}$$

At nodes P_3 and P_4, equation (3.1) reads

$$\begin{aligned} -k(\mathbf{n}_1 \cdot \mathbf{u}_1)\mathbf{n}_1 + ka\mathbf{n}_1 - k'(e-b)\mathbf{v} &= k'(a\mathbf{v} + b\mathbf{n}_1 + c\mathbf{w}_1) \\ -k(\mathbf{n}_2 \cdot \mathbf{u}_2)\mathbf{n}_2 + kd\mathbf{n}_2 + k'(e-b)\mathbf{v} &= k'(d\mathbf{v} + e\mathbf{n}_2 + f\mathbf{w}_2) \end{aligned} \tag{3.9}$$

from which we deduce $b - e = a = -d = \frac{K}{|K|}(\mathbf{n}_1 \cdot \mathbf{u}_1 - \mathbf{n}_2 \cdot \mathbf{u}_2)$ and $c = f = 0$. Now at terminals P_1 and P_2 equation (3.2) reads

$$\begin{aligned} \mathbf{A}_1 &= k(\mathbf{n}_1 \cdot \mathbf{u}_1)\mathbf{n}_1 - ka\mathbf{n}_1 - \omega^2\mathbf{M}_1 \cdot \mathbf{u}_1 \\ \mathbf{A}_2 &= k(\mathbf{n}_2 \cdot \mathbf{u}_2)\mathbf{n}_2 - kd\mathbf{n}_2 - \omega^2\mathbf{M}_2 \cdot \mathbf{u}_2, \end{aligned} \tag{3.10}$$

where $\mathbf{M}_1$ and $\mathbf{M}_2$ denote the effective masses of terminals P_1, P_2 which we have not yet fixed. Then the response matrix of the network is then

$$\mathcal{W} = \begin{pmatrix} -\omega^2\mathbf{M}_1 + (|K| - K)\mathbf{n}_1 \otimes \mathbf{n}_1 & K\mathbf{n}_1 \otimes \mathbf{n}_2 \\ K\mathbf{n}_2 \otimes \mathbf{n}_1 & -\omega^2\mathbf{M}_2 + (|K| - K)\mathbf{n}_2 \otimes \mathbf{n}_2 \end{pmatrix} \tag{3.11}$$

The key feature of this response matrix is that the off-diagonal matrix is proportional to $\mathbf{n}_1 \otimes \mathbf{n}_2$. This could have been anticipated since the spring joining terminals (P_1, P_3) exerts a force on P_1 in the direction $\mathbf{n}_1$, and this force can only depend on $\mathbf{u}_2$ through the component of $\mathbf{u}_2$ in the direction $\mathbf{n}_2$ of the spring joining terminals (P_2, P_4).

Example 9 *Considering in the previous example the particular case $\mathbf{n}_1 = \mathbf{n}_2 = \mathbf{n}$ and choosing the appropriate values for the tensors $\mathbf{M}_1$ and $\mathbf{M}_2$ we get*

$$\mathcal{W} = \begin{pmatrix} -K\mathbf{n} \otimes \mathbf{n} & K\mathbf{n} \otimes \mathbf{n} \\ K\mathbf{n} \otimes \mathbf{n} & -K\mathbf{n} \otimes \mathbf{n} \end{pmatrix}, \tag{3.12}$$

which is similar to the response matrix of a single spring but where the constant K can be negative. More important is the fact that in this structure the direction of action $\mathbf{n}$ is no longer correlated with the direction of the vector $\mathbf{x}_2 - \mathbf{x}_1$.

Remember however that we have the restriction that $\mathbf{n}$ cannot be in the direction $\mathbf{x}_2 - \mathbf{x}_1$. But we can get rid of this restriction by considering two copies of this structure: one of these copies joins terminal P_1 to an internal node P_3 placed at a point $\mathbf{x}_3$ such that $\mathbf{n}_{1,3}$ is not parallel to $\mathbf{x}_2 - \mathbf{x}_1$ while the other one joins terminal P_2 to P_3. In both copies the constant is $2K$ and the direction of action is $\mathbf{n} = \mathbf{n}_{1,2}$. It is easy to check that the response matrix of such a structure is still given by (3.12). In that way we actually get a virtual spring with possibly negative spring constant.

This makes free the position of the internal nodes: indeed, in any network, we can change the position $\mathbf{x}_r$ of a node P_r to any other position $\mathbf{x}'_r$, replacing all the springs joining P_r to other nodes P_s by a copy of Example 9 with $\mathbf{n}_1 = \mathbf{n}_{r,s}$. Clearly the response matrix will remain unchanged. Thus we have

Remark 3 *Any network has an equivalent network the internal nodes of which avoid a given finite set of points.*

Combining Remark 3 with Remark 1 enables us to assume when considering two different networks that they do not share any internal node and that the springs of the different networks do not intersect. Then we have

Remark 4 *Superposition principle : In three dimensions, if $\mathcal{W}_1$ and $\mathcal{W}_2$ are two realizable response matrices each associated with n-terminals in the same positions $P_1, P_2, \ldots, P_n$, then the response matrix $\mathcal{W}_1 + \mathcal{W}_2$ is also realizable. The network which realizes the matrix $\mathcal{W}_1 + \mathcal{W}_2$ is just a superposition of suitable modifications of the networks which realize $\mathcal{W}_1$ and $\mathcal{W}_2$.*

Example 10 *Choosing in Example 8 the appropriate values for the tensors $\mathbf{M}_1$ or $\mathbf{M}_2$ we can get*

$$\mathcal{W} = \begin{pmatrix} 0 & K\mathbf{n}_1 \otimes \mathbf{n}_2 \\ K\mathbf{n}_2 \otimes \mathbf{n}_1 & 0 \end{pmatrix} \tag{3.13}$$

As any matrix $\mathbf{W}$ is the sum of rank one matrices, the superposition principle implies that, for any matrix $\mathbf{W}$, there exists a two-terminal network the response matrix of which is

$$\mathcal{W} = \begin{pmatrix} 0 & \mathbf{W} \\ \mathbf{W}^T & 0 \end{pmatrix}. \tag{3.14}$$

And we obtain the following

Theorem 4 *In three dimensions given any set of n points $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n$, and any real $n \times n$ matrix $\mathcal{W}$ with second order tensor entries $\mathbf{W}_{ij}$ satisfying the symmetry properties (3.5), then there is purely elastic network with terminals $P_1, P_2, \ldots P_n$ at positions $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n$ and realizing $\mathcal{W}$ as its response matrix.*

Proof: It is enough to attach to each pair (P_r, P_s) a copy of Example 10 in which $\mathbf{W}$ is chosen to be $\mathbf{W}_{r,s}$, then to endow each terminal P_r with the effective mass corresponding to the symmetric matrix $\mathbf{W}_{r,r}$. Then we conclude using the superposition principle. $\square$

3.2 Elastodynamic networks with damping

An elastodynamic network with damping is a network with point masses at the nodes and viscoelastic springs joining the nodes. If we allow viscous damping in the springs, the constant k in (3.3) becomes complex with a non-positive imaginary part. (The real part of k is still non-negative, and masses are still non-negative reals.) Then the matrix $\mathcal{W}$ is complex and symmetric, with negative semidefinite imaginary part,

$$\mathrm{Imag}\mathcal{W} \leq 0, \tag{3.15}$$

which reflects the second law of thermodynamics that averaged over time the network can transform mechanical energy into heat, but not the reverse. The proof of (3.15) is similar to the electrical case.

Let us revisit the examples we gave in the previous section. Examples 5 and 6 are unchanged: the constant k in the response matrix is now complex with a positive real part and a negative imaginary part. Example 7 is still valid : indeed for any complex μ with positive imaginary part the constant k defined by $k = \frac{\mu|\mu|}{2\mu-|\mu|}\omega^2$ has a negative imaginary part and a positive real part. Then remark 2 can be generalized in

Remark 5 *Any node can be endowed by any complex symmetric tensor effective mass provided that its imaginary part is positive semidefinite.*

Theorem 5 *In three dimensions, given any set of n points $\mathbf{x}_1, \mathbf{x}_2, \dots \mathbf{x}_n$, and any complex matrix $\mathcal{W}$ with second order tensor entries $\mathbf{W}_{r,s}$ satisfying the symmetry properties (3.5) and the constraint (3.15), then there is an elastodynamic network with damping realizing $\mathcal{W}$ as its response matrix,*

Proof: Let us consider the real and imaginary parts of $\mathcal{W} = \mathcal{W}^{re} + i\mathcal{W}^{im}$. They are $n \times n$ matrices with 3×3 real entries denoted respectively $\mathbf{W}^{re}_{r,s}$ and $\mathbf{W}^{im}_{r,s}$ and can be identified with $3n \times 3n$ symmetric matrices with real coefficients. Thus we introduce the $3n$ eigenvalues $(k^m)_{m=1}^{3n}$ of $\mathcal{W}^{im}$. As $\mathcal{W}^{im}$ is a negative semidefinite symmetric matrix these eigenvalues (k^m) are non-positive reals, and the corresponding eigenvectors can be chosen to be an orthonormal set. We denote by $(\mathbf{a}^m_1, \mathbf{a}^m_2, \dots \mathbf{a}^m_n)$ a $3n$-component eigenvector (identified with an n-entry vector where the entries are 3-component vectors) associated with the eigenvalue k^m and we introduce an extra unit vector $\mathbf{b}$.

Owing to Theorem 4 we know that there exists a (no damping) elastic network with $4n$ terminals with the real response matrix $\widetilde{\mathcal{W}}$ with entries $\widetilde{\mathbf{W}}_{r,s}$ defined by

$$\widetilde{\mathbf{W}}_{r,s} = \mathbf{W}^{re}_{r,s}, \text{ if } r \leq n \text{ and } s \leq n, \qquad \widetilde{\mathbf{W}}_{r,s} = 0, \text{ if } r > n \text{ and } s > n, \tag{3.16}$$

$$\widetilde{\mathbf{W}}_{r,n+m} = \widetilde{\mathbf{W}}^T_{n+m,r} = -k^m\, \mathbf{a}^m_r \otimes \mathbf{b},\ \text{if } r \leq n \text{ and } m \in \{1, \dots 3n\}. \tag{3.17}$$

Then, owing to Remark 5, let us now endow each terminal P_{n+m} (for $m \in \{1, \dots 3n\}$) with the effective mass tensor $\mathbf{M}_{n+m} = -ik^m\omega^{-2}\,\mathbf{I}$ and consider all these terminals as internal nodes. At each node P_{n+m} (for $m \in \{1, \dots 3n\}$) we have

$$k^m \sum_{r=1}^{n} (\mathbf{a}^m_r \cdot \mathbf{u}_r)\, \mathbf{b} = -\omega^2 \mathbf{M}_{n+m} \cdot \mathbf{u}_{n+m} = ik^m \mathbf{u}_{n+m} \tag{3.18}$$

and, at each terminal P_r, for $r \leq n$:

$$\mathbf{A}_r = \sum_{s=1}^{n} \mathbf{W}^{re}_{r,s} \cdot \mathbf{u}_s + \sum_{m=1}^{3n} -k^m (\mathbf{b} \cdot \mathbf{u}_{n+m}) \mathbf{a}^m_r. \tag{3.19}$$

Using the first equation to eliminate the terms involving $\mathbf{u}_{n+m}$ in the second equation, we get

$$\mathbf{A}_r = \sum_{s=1}^{n} \left(\mathbf{W}^{re}_{r,s} + i \sum_{m=1}^{3n} \left(k^m (\mathbf{a}^m_r \otimes \mathbf{a}^m_s)\right) \right) \cdot \mathbf{u}_s. \tag{3.20}$$

Thus we get the desired response matrix. □

3.3 Planar elastodynamic networks

As in the electrical case, in two-dimensions, we have the important topological restriction that no two edges are allowed to cross without intersecting at a common node. Now we have the additional restriction that a spring between two nodes must lie along the segment joining those two nodes. Despite these restrictions we have:

Theorem 6 *Theorems 4 and 5 still hold true for planar networks.*

Proof: Example 6 can be adapted to the planar case : the simplexes we used are now simply triangles. However Remark 1 is no longer valid. Due to the topological restrictions, Example 6 cannot be used to avoid crossings. It still can be used to change the direction of the springs in a network. So we can assume that any crossing point is a generic one, which means that only two springs are crossing at that point and the angle they make is non-zero.

Let us allow, for a while, springs to intersect without interacting and let us call *pseudo-planar* such networks. In this setting, Example 7 and 8 and the superposition principle are still valid. Nothing is changed from the three dimensional case : we can construct a pseudo-planar network with any desired response matrix. Owing to the previous remark we can assume that all crossing points in this pseudo planar network are generic ones.

Now let us consider two crossing springs (let us say connecting (P_1, P_2) and connecting (P_3, P_4) with constants $k_{1,2}$ and $k_{3,4}$) in this network and let us replace these two springs by the following network:

Example 11 *Let $k_{1,2}$ and $k_{3,4}$ be any positive reals (or complex with positive real part and negative imaginary part) and consider the 4-terminal network where the terminals P_i $(i \leq 4)$ are placed at points $\mathbf{x}_i$ such that the segments $[\mathbf{x}_1, \mathbf{x}_2]$ and $[\mathbf{x}_3, \mathbf{x}_4]$ have an intersection at a single point $\mathbf{x}_5$. The network has an internal node P_5 at point $\mathbf{x}_5$. We assume that the nodes have no mass and that four springs join P_1, P_2, P_3, P_4 to P_5 with constants respectively equal to $k_{1,5} = k_{2,5} := 2k_{1,2}$ and $k_{3,5} = k_{4,5} := 2k_{3,4}$. We have*

$$\mathbf{A}_1 = \mathbf{F}_{1,5} = -2k_{1,2}(\mathbf{n}_{1,2} \otimes \mathbf{n}_{1,2}) \cdot (\mathbf{u}_1 - \mathbf{u}_5), \quad \mathbf{A}_2 = \mathbf{F}_{2,5} = -2k_{1,2}(\mathbf{n}_{1,2} \otimes \mathbf{n}_{1,2}) \cdot (\mathbf{u}_2 - \mathbf{u}_5), \tag{3.21}$$

$$\mathbf{A}_3 = \mathbf{F}_{3,5} = -2k_{3,4}(\mathbf{n}_{3,4} \otimes \mathbf{n}_{3,4}) \cdot (\mathbf{u}_3 - \mathbf{u}_5), \quad \mathbf{A}_4 = \mathbf{F}_{4,5} = -2k_{3,4}(\mathbf{n}_{3,4} \otimes \mathbf{n}_{3,4}) \cdot (\mathbf{u}_4 - \mathbf{u}_5), \tag{3.22}$$

$$\mathbf{F}_{1,5} + \mathbf{F}_{2,5} + \mathbf{F}_{3,5} + \mathbf{F}_{4,5} = 0. \tag{3.23}$$

Owing to the geometrical assumptions $(\mathbf{n}_{1,2}, \mathbf{n}_{3,4})$ makes a basis and we introduce its dual basis $(\mathbf{n}^*_{1,2}, \mathbf{n}^*_{3,4})$. Writing $\mathbf{u}_5 = a\mathbf{n}^*_{1,2} + b\mathbf{n}^*_{3,4}$, the previous system of equations becomes

$$\mathbf{A}_1 = -2k_{1,2}(\mathbf{n}_{1,2} \cdot \mathbf{u}_1 - a)\mathbf{n}_{1,2}, \quad \mathbf{A}_2 = -2k_{1,2}(\mathbf{n}_{1,2} \cdot \mathbf{u}_2 - a)\mathbf{n}_{1,2} \tag{3.24}$$

$$\mathbf{A}_3 = -2k_{3,4}(\mathbf{n}_{3,4} \cdot \mathbf{u}_3 - b)\mathbf{n}_{3,4}, \quad \mathbf{A}_4 = -2k_{3,4}(\mathbf{n}_{3,4} \cdot \mathbf{u}_4 - b)\mathbf{n}_{3,4} \tag{3.25}$$

$$\mathbf{n}_{1,2} \cdot (\mathbf{u}_1 + \mathbf{u}_2) = 2a, \quad \mathbf{n}_{3,4} \cdot (\mathbf{u}_3 + \mathbf{u}_4) = 2b \tag{3.26}$$

The elimination of $\mathbf{u}_5$ (i.e. of a and b) in this system leads to

$$\mathbf{A}_1 = -\mathbf{A}_2 = k_{1,2}(\mathbf{n}_{1,2} \otimes \mathbf{n}_{1,2}) \cdot (\mathbf{u}_1 - \mathbf{u}_2), \qquad \mathbf{A}_3 = -\mathbf{A}_4 = k_{3,4}(\mathbf{n}_{3,4} \otimes \mathbf{n}_{3,4}) \cdot (\mathbf{u}_3 - \mathbf{u}_4) \tag{3.27}$$

The response matrix of this network is equivalent to the response of two springs joining directly and independently P_1 to P_2 and P_3 to P_4 with constants $k_{1,2}$ and $k_{3,4}$.

Replacing two crossing springs by a copy of Example 11 removes a crossing point (and does not create any new one). Hence we can successively remove all crossing points in the pseudo planar network and obtain a true planar network with the desired response matrix. This analysis is valid in both purely elastic and damping cases. □

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